ALGEBRA COMPREHENSIVE EXAMINATION

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Directions: *Answer 5 questions only.* You must answer *at least one* from each of linear algebra, groups, and synthesis. Indicate CLEARLY which problems you want us to grade—otherwise, we will select which ones to grade, and they may not be the ones that you want us to grade. Be sure to show enough work that your answers are adequately supported.

Notation: $\mathbb{R}$ is the set of real numbers; $\mathbb{Z}$ is the set of integers; $\mathbb{C}$ is the set of complex numbers; $GL(V)$ is the group of invertible linear maps from a vector space V to itself; $GL_n(F)$ is the group of all invertible $n \times n$ matrices with entries in the field F; $SO_n(F)$ is the subgroup of $GL_n(F)$ consisting of all matrices that are orthogonal and have determinant one.

Linear Algebra

(1) Let $T : V \to V$ be a linear transformation from a real vector space V to itself. Let u_1 and u_2 be nonzero vectors in V such that $T(u_1) = u_1$ and $T(u_2) = 2u_2$. Show that $\{u_1, u_2\}$ is linearly independent. *Answer: Suppose that $c_1u_1 + c_2u_2 = 0$ for some $c_1, c_2 \in \mathbb{R}$. Then $0 = T(0) = T(c_1u_1 + c_2u_2) = c_1T(u_1) + c_2T(u_2) = c_1u_1 + 2c_2u_2$. We now have linear equations $c_1u_1 + c_2u_2 = 0$ and $c_1u_1 + 2c_2u_2 = 0$. These can be solved to give $c_1u_1 = 0$ and $c_2u_2 = 0$. Since u_1 and u_2 are nonzero, we get $c_1 = c_2 = 0$.*

(2) (a) Prove that every matrix with entries from $\mathbb{C}$ has at least one eigenvalue in $\mathbb{C}$.
(b) Give an example of a matrix with entries from $\mathbb{R}$ that does not have an eigenvalue in $\mathbb{R}$.

Answer: (a) Let A be a matrix with entries from $\mathbb{C}$. Let $f(x) = \det(A - xI)$ be the characteristic polynomial of A. By the fundamental theorem of algebra, every polynomial with coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$, so there exists $\lambda \in \mathbb{C}$ such that $f(\lambda) = 0$. This λ is an eigenvalue of A.

(b) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has no eigenvalues in $\mathbb{R}$. We can see this geometrically: the matrix is a quarter-turn rotation of $\mathbb{R}^2$, and no non-zero vector in $\mathbb{R}^2$ is a scalar multiple of its quarter-turn rotation. We can also see this algebraically: the characteristic polynomial is $1 + x^2$, which does not have a zero in $\mathbb{R}$.

(3) Let $V = M_n(\mathbb{R})$ be the vector space of $n \times n$ real matrices. Let I be the identity $n \times n$ matrix. Prove that for all matrices $A, B \in V$ the equation $A \cdot B - B \cdot A = I$ is impossible.

Answer: Need to prove
(i) the trace function $tr : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$ given by

$$tr(A) = \text{ the trace of } A$$

is linear (that is, it is a linear functional),
(ii) $tr(A \cdot B) = tr(B \cdot A)$, and
(iii) $tr(A \cdot B - B \cdot A) = tr(A \cdot B) - tr(B \cdot A) = 0$, but $tr(I) \neq 0$.

Groups

(1) Let H and K be subgroups of a group G such that $G = HK$. Show that the following are equivalent:

(a) $H \cap K = \{e\}$
(b) Each element of $g \in G$ can be written uniquely in the form $g = hk$ with $h \in H$ and $k \in K$.

Answer: Suppose that $H \cap K = \{e\}$ and $g \in G$. Since $G = HK$, we have $g = hk$ for some $h \in H$ and $k \in K$, and it remains only to prove uniqueness.

Suppose that $g = h_1k_1 = h_2k_2$ with $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Then $h_2^{-1}h_1 = k_2k_1^{-1}$ is an element of $H \cap K$. By assumption, $h_2^{-1}h_1 = k_2k_1^{-1} = e$, and so $h_2 = h_1$ and $k_2 = k_1$.

Conversely, suppose that (b) holds and $g \in H \cap K$. Then $g = ge$ with $g \in H$ and $e \in K$ and also $g = eg$ with $e \in H$ and $g \in K$, By the uniqueness of such expressions, $g = e$.

(2) (a) Prove or disprove: For every finite group G, every element $g \in G$ has finite order.
(b) Prove or disprove: For every infinite group G, every non-identity element $g \in G$ has infinite order.

Answer: (a) The set $\{g, g^2, g^3, \ldots\}$, being a subset of the finite group G, is finite. Let n be its cardinality; then $g, \ldots, g^{n+1}$ cannot be $n+1$ distinct elements, so there exist i, j such that $i < j$ and $g^i = g^j$. Thus $g^{j-i} = 1$, so g has finite order — its order is at most $j - i$.

(b) There are many counterexamples, for instance $\mathbb{Z} \times \mathbb{Z}_2$ (under addition) in which $(0, 1)$ has order 2.

(3) Let G be a group. Recall that an *automorphism* of G is an isomorphism from G to G.
(a) Fix $a \in G$, and define a map $\phi_a \colon G \to G$ by $\phi_a(g) = aga^{-1}$. Prove that ϕ_a is an automorphism of G.
(b) Suppose H is a subgroup of G with the property that, for every automorphism α of G, we have $\alpha(H) = H$. Prove that H is a normal subgroup of G.

Answer: (a) If $g, h \in G$, then $\phi_a(gh) = agha^{-1} = (aga^{-1})(aha^{-1}) = \phi_a(g)\phi_a(h)$, so ϕ_a is a homomorphism. And ϕ_a is invertible because its inverse is $\phi_{a^{-1}}$.

(b) For all $a \in G$, we have $\phi_a(H) = H$ since ϕ_a is an automorphism. Thus $aHa^{-1} = H$ for all $a \in G$, so H is normal.

Synthesis

(1) Let F be a field, and define $F^* = F \setminus \{0\}$. It is a fact (you do not have to prove) that F^* is a group under multiplication. Let V be a finite-dimensional vector space over F. For each $c \in F^*$, define $\phi_c \colon V \to V$ by

$$\phi_c(v) = c \cdot v$$

(i.e., ϕ_c is scalar multiplication by c).
(a) For fixed c, prove that ϕ_c is an invertible linear map.
(b) Prove that the map $\Phi \colon F^* \to GL(V)$ given by $\Phi(c) = \phi_c$ is a group homomorphism.

Answer: (a) $\phi_c(v + w) = c \cdot (v + w) = c \cdot v + c \cdot w = \phi_c(v) + \phi_c(w)$.
$\phi_c(a \cdot v) = c \cdot (a \cdot v) = a \cdot (c \cdot v) = a \cdot \phi_c(v)$.
That shows that ϕ_c is linear, and ϕ_c is invertible because its inverse is $\phi_{(c^{-1})}$.
(b) $\phi_{cd}(v) = (cd) \cdot v = c \cdot (d \cdot v) = \phi_c(\phi_d(v))$. Thus $\phi_{cd} = \phi_c \circ \phi_d$.

(2) Let $G = \left\{ \begin{pmatrix} 1+n & -n \\ n & 1-n \end{pmatrix} \mid n \in \mathbb{Z} \right\}$.

(a) Prove that G is group under matrix multiplication.

(b) Prove that G and $(\mathbb{Z}, +)$ are isomorphic.

Answer: (a) Verify that

$$\begin{pmatrix} 1+n & -n \\ n & 1-n \end{pmatrix} \cdot \begin{pmatrix} 1+m & -m \\ m & 1-m \end{pmatrix} = \begin{pmatrix} 1+n+m & -(n+m) \\ n+m & 1-(n+m) \end{pmatrix},$$

and therefore the multiplication is closed in G. It's relatively easy to check the operation is associative (and this is true for matrix multiplication anyway. $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity element. And $\begin{pmatrix} 1+n & -n \\ n & 1-n \end{pmatrix}^{-1} = \begin{pmatrix} 1-n & n \\ -n & 1+n \end{pmatrix}$. So, G is a group.

Answer: (b) Define $\phi : \mathbb{Z} \to G$ by $\phi(n) = \begin{pmatrix} 1+n & -n \\ n & 1-n \end{pmatrix}$. Part of the work in (a) shows that ϕ is an homomorphism. It's obviously onto and $Ker(\phi) = \{0\}$. Thus G and $\mathbb{Z}$ are isomorphic.

(3) Let G be the subgroup of $GL_3(\mathbb{R})$ generated by the matrices

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

(a) Is G contained in $SO_3(\mathbb{R})$? Explain. *Answer: Yes. Since* $\det A = \det B = 1$ *and $AA^t = BB^t = I$, both A and B are in $SO_3(\mathbb{R})$.*

(b) Is G abelian? Explain. *Answer: No. For example, $AB \neq BA$.*

(c) Find an element of G with order two. *Answer: For example, $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ has order two.*

(d) Explain why G must have order 12 or greater. *Answer: For example, since G has elements of order 2 and 3, $|G|$ is a multiple of 6. The only nonabelian group of order 6, S_3, has two elements of order 3 whereas G has at least four elements of order three, namely, A, A^2, B and B^2. (In fact, $G \cong A_4$ and has order 12.)*